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Asymptotic Behavior of Sobolev-Type Orthogonal Polynomials on the Unit Circle*

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We study the asymptotic behavior of the sequence of polynomials orthogonal with respect to the discrete Sobolev inner product on the unit circle

$$\langle f, g\rangle = \int_0^{2\pi} f(e^{i\theta})\, \overline{g(e^{i\theta})}\, d\mu(\theta) + f(\mathbf{Z})\, Ag(\mathbf{Z})^H,$$

where $f(\mathbf{Z}) = (f(z_1), ..., f^{(l_1)}(z_1), ..., f(z_m), ..., f^{(l_m)}(z_m))$, A is a $M \times M$ positive definite matrix or a positive semidefinite diagonal block matrix, $M = l_1 + \cdots + l_m + m$, $d\mu$ belongs to a certain class of measures, and $|z_i| > 1$, $i = 1, 2, ..., m$. © 1999 Academic Press

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1. INTRODUCTION AND STATEMENTS OF RESULT

In the past few years, there has been a growing interest in the study of nonstandard inner products and the properties of the orthogonal polynomials which they generate. Among these, Sobolev-type inner products and the corresponding Sobolev-type orthogonal polynomials are of particular interest. As in the classical theory of orthogonal polynomials, the asymptotic behavior of sequences of Sobolev-type orthogonal polynomials plays a central role in questions related to their application in approximation processes, in particular, in Fourier expansions.

This paper is devoted to the study of the asymptotic properties of the so-called discrete Sobolev-type orthogonal polynomials on the unit circle.

Let μ be a probability measure whose support consists of an infinite set of points contained in $[0, 2\pi]$. Let $\{\varphi_n\}_{n\geqslant 0}$, $\varphi_n(z)=k_n z^n +$ lower degree terms, $k_n>0$, be the sequence of orthonormal polynomials with respect to μ In all that follows we assume that $\lim_{n\to\infty} \varphi_n(0)/k_n=0$, and denote this by $\mu\in\mathcal{N}$ (μ belongs to Nevai's class of measures). A well-known result of Rakhmanov [10] states that if $\mu'>0$ a.e. on $[0, 2\pi]$ then $\mu\in\mathcal{N}$. Along with the sequence of orthonormal polynomials $\{\varphi_n\}_{n\geqslant 0}$, we consider the sequence $\{\varphi_n^*\}_{n\geqslant 0}$ of the reversed polynomials, which as usual are defined by $\varphi_n^*(z)=z^n\overline{\varphi_n(1/\bar{z})}$.

DEFINITION 1. Let μ be a probability measure with an infinite subset of the interval $[0, 2\pi]$ as its support. A discrete Sobolev inner product on the unit circle is given by

$$\langle f, g\rangle=\int_0^{2\pi} f(e^{i\theta})\,\overline{g(e^{i\theta})}\,d\mu(\theta)+f(\mathbf{Z})\,Ag(\mathbf{Z})^H, \tag{1}$$

where $f(\mathbf{Z})=(f(z_1), ..., f^{(l_1)}(z_1), ..., f(z_m), ..., f^{(l_m)}(z_m))$, A is an $M\times M$ positive semi-definite matrix, $M=l_1+\cdots+l_m+m$, $|z_i|>1$, $i=1, 2, ..., m$ and $g(\mathbf{Z})^H$ denotes the conjugate transpose of the vector $g(\mathbf{Z})$.

Since A is positive semi-definite, the inner product $\langle\cdot,\cdot\rangle$ is positive definite. Therefore, there exists a sequence $\{\psi_n\}_{n\geqslant 0}$, $\psi_n(z)=\gamma_n z^n+$ lower degree terms, $\gamma_n>0$, which is orthonormal with respect to (1). We are interested in the asymptotic behavior of the sequence of ratios $\{\psi_n/\varphi_n\}_{n\geqslant 0}$, commonly called relative asymptotics of ψ_n with respect to φ_n. We will show that if $\mu\in\mathcal{N}$ and A is positive definite, then there is relative asymptotics (see (2) below). Since for $\mu\in\mathcal{N}$ the sequence $\{\varphi_n\}_{n\geqslant 0}$ is known to have ratio asymptotics, one immediately derives ratio asymptotics for the sequence $\{\psi_n\}_{n\geqslant 0}$ (see (4)) as well as other types of asymptotic relations (see (5)).

Similar results have been obtained for the case when the measure μ is supported on a interval of the real line. We wish to refer to several papers in this setting from which we have borrowed some ideas. In [8], a very simple case of Sobolev orthogonal polynomials on the real line is considered in which the discrete part has one point and only the first derivative appears. This paper contains a very nice algebraic technique which we have adapted for our purpose. The results of [8] were substantially improved in [6], the results of which are comparable in generality with the ones exhibited in this paper for the case of the unit circle. Our paper combines ideas from [6] and [8] but remains closer to [8] in the sense that greater emphasis is placed in the use of the kernel function in order to derive appropriate algebraic relations to deal with the connection between the polynomials ψ_n and the φ_n. The analogue of some determinantal expressions which appear in [1] have also been very useful for us.

Discrete Sobolev-type orthogonal polynomials on the unit circle have also been studied before. In [2], the case when $m=1$, $l_1=1$, $|z_1|=1$; and $\mu \in \mathcal{N}$ was treated. In [5], the authors consider m different points but only first derivative in the discrete part.

In the following the symbol $\rightrightarrows$ means uniform convergence on compact subsets of the indicated region. We prove:

THEOREM 1. *Consider an inner product of type (1) such that $\mu \in \mathcal{N}$ and the matrix A is positive definite. It holds*

$$\frac{\psi_n^{(k)}(z)}{\varphi_n^{(k)}(z)} \rightrightarrows \prod_{i=1}^{m} \left(\frac{\overline{z}_i (z - z_i)}{|z_i| \, (z\overline{z}_i - 1)} \right)^{l_i+1}, \qquad |z|>1, \quad k=0, 1, ..., \tag{2}$$

$$\frac{\psi_n(z)}{\varphi_n^*(z)} \rightrightarrows 0, \qquad |z|<1,$$

$$\lim_{n\to\infty} \frac{k_n}{\gamma_n} = \prod_{i=1}^{m} |z_i|^{l_i+1}. \tag{3}$$

An immediate consequence of Theorem 1 is

COROLLARY 1. *On the region $\{z \in \mathbb{C}: |z|>1\} \backslash \{z_j\}_{j=1}^{m}$, we have*

$$\frac{\psi_{n+1}^{(k)}(z)}{\psi_n^{(k)}(z)} \rightrightarrows z, \tag{4}$$

$$\frac{\psi_n^{(k+1)}(z)}{n\psi_n^{(k)}(z)} \rightrightarrows \frac{1}{z}, \tag{5}$$

for $k=0, 1, ...$

Remark. Notice that (3) follows from (2) if we make $z \to \infty$, but in the proof of Theorem 1 we deduce first (3), and then we use this information to get (2).

2. NOTATION AND BASIC TOOLS ABOUT ASYMPTOTIC PROPERTIES

Following the notation introduced in definition 1, if

$$\mathbf{Z} = (\underbrace{z_1, ..., z_1}_{l_1+1}, ..., \underbrace{z_m, ..., z_m}_{l_m+1})$$

then

$$f(\mathbf{Z}) = (f(z_1), f'(z_1), ..., f^{(l_1)}(z_1), ..., f(z_m), f'(z_m), ..., f^{(l_m)}(z_m)).$$

Let μ be a probability measure whose support contains infinitely many points of the interval $[0, 2\pi]$ as its support. Assume that $\mu \in \mathcal{N}$ and let $\{\varphi_n\}_{n \geq 0}$, $\varphi_n(z) = k_n z^n +$ lower degree terms, $k_n > 0$, be the sequence of orthonormal polynomials with respect to this measure. Let

$$K_n(z, \eta) = \sum_{k=0}^{n-1} \varphi_k(z) \overline{\varphi_k(\eta)}$$

be the kernel polynomials associated to μ. Then

$$K_n^{(i,j)}(z, \eta) = \sum_{k=0}^{n-1} \varphi_k^{(i)}(z) \overline{\varphi_k^{(j)}(\eta)}.$$

It is very well known (cf. [10]) that

$$\frac{\varphi_{n+1}(z)}{\varphi_n(z)} \rightrightarrows z, \qquad |z| > 1,$$

and using the same technique as in the proof of Lemma 1 below, we get

$$\frac{\varphi_{n+1}^{(k)}(z)}{\varphi_n^{(k)}(z)} \rightrightarrows z, \qquad |z| > 1, \quad k = 0, 1, ..., \tag{6}$$

$$\frac{\varphi_n^{(k+1)}(z)}{n\varphi_n^{(k)}(z)} \rightrightarrows \frac{1}{z}, \qquad |z| > 1, \quad k = 0, 1, ... \tag{7}$$

We also point out the following result that can be found in [9] Theorem 4,

$$\frac{\varphi_n^*(z)}{\varphi_n(z)} \rightrightarrows 0, \qquad |z|>1, \tag{8}$$

or equivalently

$$\frac{\varphi_n(z)}{\varphi_n^*(z)} \rightrightarrows 0, \qquad |z|<1. \tag{9}$$

Now, we include some auxiliary results.

LEMMA 1. *If* $\mu \in \mathcal{N}$ *then*

$$\frac{K_n^{(i,j)}(z,\xi)}{\varphi_n^{(i)}(z)\,\overline{\varphi_n^{(j)}(\xi)}} \rightrightarrows \frac{1}{z\bar{\xi}-1}, \qquad |z|, |\xi|>1, \quad i, j=0, 1, ...$$

Proof. First, from (7), we have

$$\frac{\varphi_n^{(q)}(z)}{\varphi_n^{(p)}(z)} \rightrightarrows 0, \qquad |z|>1, \quad p>q \geqslant 0. \tag{10}$$

We claim that

$$\frac{\varphi_n^{*(q)}(z)}{\varphi_n^{(p)}(z)} \rightrightarrows 0, \qquad |z|>1, \quad p \geqslant q \geqslant 0. \tag{11}$$

By using (6), we only need to prove

$$\frac{\varphi_n^{*(p)}(z)}{\varphi_n^{(p)}(z)} \rightrightarrows 0, \qquad |z|>1. \tag{12}$$

For $p=1$, we have (8). We proceed by induction; let us assume that (12) holds for $p=k$ and let us prove that (12) also holds for $p=k+1$. In fact, since

$$\frac{\varphi_n^{*(k+1)}(z)}{\varphi_n^{(k+1)}(z)} = \frac{\varphi_n^{(k)}(z)}{\varphi_n^{(k+1)}(z)} \left(\frac{\varphi_n^{*(k)}(z)}{\varphi_n^{(k)}(z)} \right)' + \frac{\varphi_n^{*(k)}(z)}{\varphi_n^{(k)}(z)},$$

using (6) and (10), we deduce that for $p=k+1$ the result is also true.

Next, notice that for $s, t=0, 1, ...,$

$$K_n^{(s,t)}(z,w)=\overline{\frac{\partial^t}{\partial w^t}\,\overline{\frac{\partial^s}{\partial z^s}\left(\frac{\varphi_n^*(z)\,\overline{\varphi_n^*(w)}-\varphi_n(z)\,\overline{\varphi_n(w)}}{1-\bar{w}z}\right)}}$$

$$=\sum_{l=0}^{s}\sum_{r=0}^{t}\binom{t}{r}\binom{s}{l}\{\varphi_n^{*(l)}(z)\,\overline{\varphi_n^{*(r)}(w)}-\varphi_n^{(l)}(z)\,\overline{\varphi_n^{(r)}(w)}\}$$

$$\times\overline{\frac{\partial^{t-r}}{\partial w^{t-r}}\,\overline{\frac{\partial^{s-l}}{\partial z^{s-l}}\frac{1}{1-\bar{w}z}}}. \tag{13}$$

Thus the lemma follows from (10), (11), and (13). ∎

COROLLARY 2. *If $\mu\in\mathcal{N}$ then*

$$\frac{K_n^{(i,j)}(z,\xi)}{\varphi_n^{(p)}(z)\,\overline{\varphi_n^{(q)}(\xi)}}\rightrightarrows 0, \qquad |z|, |\xi|>1$$

for $p\geqslant i$, $q>j$ or $p>i$, $q\geqslant j\geqslant 0$.

LEMMA 2. *If $\mu\in\mathcal{N}$ then*

$$\frac{K_n^{(0,j)}(z,\xi)}{\varphi_n^*(z)\,\overline{\varphi_n^{(j)}(\xi)}}\rightrightarrows 0, \qquad |z|<1, |\xi|>1, \quad j=0, 1, ...$$

Proof. This result easily follows from (9) and (13). ∎

LEMMA 3. *If $\mu\in\mathcal{N}$, we have*

$$\frac{1}{\varphi_n^{(i)}(z)}\rightrightarrows 0, \; |z|>1, \qquad i=0, 1, ...$$

Proof. It is a straightforward consequence of (6). ∎

LEMMA 4. *Let Q be an $M\times M$ nonsingular matrix, and u, x two M-column vectors. The following identity holds*:

$$1-x^TQ^{-1}u=\frac{\det[Q-ux^T]}{\det Q}.$$

Proof. We consider the matrix identities

$$\begin{pmatrix} Q & u\\ x^T & 1\end{pmatrix}\begin{pmatrix} I_M & -Q^{-1}u\\ 0_{1\times M} & 1\end{pmatrix}=\begin{pmatrix} Q & 0_{M\times 1}\\ x^T & 1-x^TQ^{-1}u\end{pmatrix}$$

$$\begin{pmatrix} I_M & -u\\ 0_{1\times M} & 1\end{pmatrix}\begin{pmatrix} Q & u\\ x^T & 1\end{pmatrix}=\begin{pmatrix} Q-ux^T & 0_{M\times 1}\\ x^T & 1\end{pmatrix},$$

where $0_{n\times m}$ denotes the zero matrix of order $n\times m$. Now taking determinants in both expressions we get the result. ∎

Let $\mathbb{K}_n$ be the $M\times M$ matrix

$$\begin{pmatrix} K_n(z_1,z_1) & \cdots & K_n^{(l_1,0)}(z_1,z_1) & \cdots & K_n(z_m,z_1) & \cdots & K_n^{(l_m,0)}(z_m,z_1) \\ K_n^{(0,1)}(z_1,z_1) & \cdots & K_n^{(l_1,1)}(z_1,z_1) & \cdots & K_n^{(0,1)}(z_m,z_1) & \cdots & K_n^{(l_m,1)}(z_m,z_1) \\ \vdots & & \vdots & & \vdots & & \vdots \\ K_n^{(0,l_1)}(z_1,z_1) & \cdots & K_n^{(l_1,l_1)}(z_1,z_1) & \cdots & K_n^{(0,l_1)}(z_m,z_1) & \cdots & K_n^{(l_m,l_1)}(z_m,z_1) \\ \vdots & & \vdots & & \vdots & & \vdots \\ K_n(z_1,z_m) & \cdots & K_n^{(l_1,0)}(z_1,z_m) & \cdots & K_n(z_m,z_m) & \cdots & K_n^{(l_m,0)}(z_m,z_m) \\ K_n^{(0,1)}(z_1,z_m) & \cdots & K_n^{(l_1,1)}(z_1,z_m) & \cdots & K_n^{(0,1)}(z_m,z_m) & \cdots & K_n^{(l_m,1)}(z_m,z_m) \\ \vdots & & \vdots & & \vdots & & \vdots \\ K_n^{(0,l_m)}(z_1,z_m) & \cdots & K_n^{(l_1,l_m)}(z_1,z_m) & \cdots & K_n^{(0,l_m)}(z_m,z_m) & \cdots & K_n^{(l_m,l_m)}(z_m,z_m) \end{pmatrix} \tag{14}$$

This matrix can be described by blocks. The r, s block is the $(l_r+1)\times(l_s+1)$ matrix

$$(K_n^{(j,i)}(z_s,\overline{z_r}))_{0=0,\,\ldots,\,l_r}^{j=0,\,\ldots,\,l_s},$$

where $r, s=1, ..., m$.

THEOREM 2. *The matrix $\mathbb{K}_n$ is positive definite for $n\geqslant M$ when $z_i\neq z_j$, $i, j=1, ..., m$.*

Proof. Let us consider the matrix

$$G:=\begin{pmatrix} \varphi_0(z_1) & \varphi_1(z_1) & \cdots & \varphi_{n-1}(z_1) \\ \varphi_0'(z_1) & \varphi_1'(z_1) & \cdots & \varphi_{n-1}'(z_1) \\ \vdots & \vdots & & \vdots \\ \varphi_0^{(l_1)}(z_1) & \varphi_1^{(l_1)}(z_1) & \cdots & \varphi_{n-1}^{(l_1)}(z_1) \\ \vdots & \vdots & & \vdots \\ \varphi_0(z_m) & \varphi_1(z_m) & \cdots & \varphi_{n-1}(z_m) \\ \varphi_0'(z_m) & \varphi_1'(z_m) & \cdots & \varphi_{n-1}'(z_m) \\ \vdots & \vdots & & \vdots \\ \varphi_0^{(l_m)}(z_m) & \varphi_1^{(l_m)}(z_m) & \cdots & \varphi_{n-1}^{(l_m)}(z_m) \end{pmatrix}.$$

Notice that

$$\mathbb{K}_n = \bar{G} G^T.$$

Using this factorization of the matrix $\mathbb{K}_n$, if we denote by x a row vector of size M, it holds

$$\bar{x} \bar{G} G^T x^T = \overline{xG} (xG)^T \geqslant 0.$$

So, in order to prove that $\mathbb{K}_n$ is a positive definite matrix it is sufficient to prove that the matrix G is non-singular. This follows from the fact that G is the matrix of a Hermite interpolation problem (expressed in the basis $\{\varphi_i\}$). ∎

Remark. We point out that in the proof above we have not used the orthogonality property of the sequence of polynomials $\{\varphi_n\}_{n \geqslant 0}$. In fact, we have only used that $\forall n \geqslant 0$, $\deg \varphi_n = n$.

Let us consider the following function $g(z, w) = 1/(zw - 1)$. We denote

$$g^{(i,j)}(z, w) := \frac{\partial^{i+j}}{\partial z^i \, \partial w^j} g(z, w).$$

Let F_m be the $M \times M$ matrix

$$\begin{pmatrix} g(z_1, \overline{z_1}) & \cdots g_1^{(l_1,0)}(z_1, \overline{z_1}) \cdots & g(z_m, \overline{z_1}) & \cdots g^{(l_m,0)}(z_m, \overline{z_1}) \\ g^{(0,1)}(z_1, \overline{z_1}) & \cdots g^{(l_1,1)}(z_1, \overline{z_1}) \cdots & g^{(0,1)}(z_m, \overline{z_1}) & \cdots g^{(l_m,1)}(z_m, \overline{z_1}) \\ \vdots & \vdots & \vdots & \vdots \\ g^{(0,l_1)}(z_1, \overline{z_1}) & \cdots g^{(l_1,l_1)}(z_1, \overline{z_1}) \cdots & g^{(0,l_1)}(z_m, \overline{z_1}) & \cdots g^{(l_m,l_1)}(z_m, \overline{z_1}) \\ \vdots & \vdots & \vdots & \vdots \\ g(z_1, \overline{z_m}) & \cdots g^{(l_1,0)}(z_1, \overline{z_m}) \cdots & g(z_m, \overline{z_m}) & \cdots g^{(l_m,0)}(z_m, \overline{z_m}) \\ g^{(0,1)}(z_1, \overline{z_m}) & \cdots g^{(l_1,1)}(z_1, \overline{z_m}) \cdots & g^{(0,1)}(z_m, \overline{z_m}) & \cdots g^{(l_m,1)}(z_m, \overline{z_m}) \\ \vdots & \vdots & \vdots & \vdots \\ g^{(0,l_m)}(z_1, \overline{z_m}) & \cdots g^{(l_1,l_m)}(z_1, \overline{z_m}) \cdots & g^{(0,l_m)}(z_m, \overline{z_m}) & \cdots g^{(l_m,l_m)}(z_m, \overline{z_m}) \end{pmatrix}. \tag{15}$$

This matrix can be described by blocks. The r, s block is an $(l_r, 1) \times (l_s + 1)$ matrix

$$(g^{(j,i)}(z_s, \overline{z_r}))_{i=0,\dots,l_r}^{j=0,\dots,l_s},$$

where $r, s = 1, \dots, m$.

THEOREM 3. *The matrix F_m defined in* (15) *is non-singular.*

Proof. Let us suppose that $|F_m| = 0$. In this case the linear dependence of the rows of the matrix F_m is equivalent to the existence of $c_{ij} \in \mathbb{C}$, $i = 1, ..., m$, $j = 0, ..., l_i$ such that the function

$$f(z) = \sum_{j=0}^{l_1} c_{1j} g^{(0,j)}(z, \overline{z_1}) + \cdots + \sum_{j=0}^{l_m} c_{mj} g^{(0,j)}(z, \overline{z_m}) \not\equiv 0$$

has at each z_i a zero of degree at least $l_i + 1$. Thus, it has at least M zeros, taking account of multiplicity. But it is immediate to check that

$$f(z) = \frac{P(z)}{Q(z)},$$

where P is a polynomial of degree at most $M - 1$ and Q is a polynomial of degree M. This leads us to a contradiction. ∎

3. PROOF OF THEOREM 1

First we deduce some algebraic expressions. Expanding ψ_n in terms of $\{\varphi_j\}_{j \geq 0}$, we have

$$\psi_n(z) = \frac{\gamma_n}{k_n} \varphi_n(z) + \sum_{k=0}^{n-1} a_{k,n} \varphi_k(z), \tag{16}$$

where

$$\begin{aligned} a_{k,n} &= \int_0^{2\pi} \psi_n(e^{i\theta}) \overline{\varphi_k(e^{i\theta})} \, d\mu(\theta) \\ &= -\psi_n(\mathbf{Z}) A \varphi_k(\mathbf{Z})^H \qquad \text{for} \quad k = 0, 1, ..., n-1. \end{aligned}$$

Substituting this expression in (16), we obtain

$$\begin{aligned} \psi_n(z) &= \frac{\gamma_n}{k_n} \varphi_n(z) - \psi_n(\mathbf{Z}) A \sum_{k=0}^{n-1} \varphi_k(\mathbf{Z})^H \varphi_k(z) \\ &= \frac{\gamma_n}{k_n} \varphi_n(z) - \psi_n(\mathbf{Z}) A K_n(z, \mathbf{Z})^T, \end{aligned} \tag{17}$$

where

$$K_n(z, \mathbf{Z}) = (K_n(z, z_1), ..., K_n^{(0, l_1)}(z, z_1), ..., K_n(z, z_m), ..., K_n^{(0, l_m)}(z, z_m)).$$

Now, we take consecutive derivatives and we substitute $z = z_1, ..., z = z_m$ in order to eliminate $\psi_n(\mathbf{Z})$. From this last expression it follows that

$$\psi_n^{(s)}(z_i) = \frac{\gamma_n}{k_n} \varphi_n^{(s)}(z_i) - \psi_n(\mathbf{Z}) \, A K_n^{(s)}(z_i, \mathbf{Z})^T$$

for $i = 1, ..., m$, $s = 0, 1, ..., l_i$. So, we get

$$\psi_n(\mathbf{Z}) = \frac{\gamma_n}{k_n} \varphi_n(\mathbf{Z}) - \psi_n(\mathbf{Z}) \, A \, \mathbb{K}_n, \tag{18}$$

where $\mathbb{K}_n$ denotes the $M \times M$ matrix defined in (14). From (18), we get

$$\psi_n(\mathbf{Z})[I_M + A \, \mathbb{K}_n] = \frac{\gamma_n}{k_n} \varphi_n(\mathbf{Z}),$$

where I_M denotes the $M \times M$ identity matrix. From Theorem 2, $\mathbb{K}_n$ is a positive definite matrix; therefore,

$$I_M + A \, \mathbb{K}_n = [\mathbb{K}_n^{-1} + A] \, \mathbb{K}_n.$$

Now, if we take into account that both $\mathbb{K}_n^{-1}$ and A are positive definite matrices, then $\mathbb{K}_n^{-1} + A$ is a positive definite matrix. Thus

$$I_M + A \, \mathbb{K}_n$$

is a non-singular matrix because it is the product of two non singular matrices. Therefore, we can write

$$\psi_n(\mathbf{Z}) = \frac{\gamma_n}{k_n} \varphi_n(\mathbf{Z})[I_M + A \, \mathbb{K}_n]^{-1}.$$

Let us substitute this expression in (17), multiply it by k_n/γ_n, and divide by $\varphi_n(z)$. Thus we obtain

$$\frac{k_n}{\gamma_n} \frac{\psi_n(z)}{\varphi_n(z)} = 1 - \varphi_n(\mathbf{Z})[I_M + A \, \mathbb{K}_n]^{-1} \, A \, \frac{K_n(z, \mathbf{Z})^T}{\varphi_n(z)}. \tag{19}$$

On the other hand, we also have

$$\langle \psi_n, \varphi_n \rangle = \int_0^{2\pi} \psi_n(e^{i\theta}) \, \overline{\varphi_n(e^{i\theta})} \, d\mu(\theta) + \psi_n(\mathbf{Z}) \, A \varphi_n(\mathbf{Z})^H,$$

$$\frac{k_n}{\gamma_n} = \frac{\gamma_n}{k_n} + \psi_n(\mathbf{Z}) \, A \varphi_n(\mathbf{Z})^H.$$

Multiplying by k_n/γ_n and substituting $\psi_n(\mathbf{Z})$, we have

$$\left(\frac{k_n}{\gamma_n}\right)^2 = 1 + \frac{k_n}{\gamma_n}\,\varphi_n(\mathbf{Z})[I_M + A\,\mathbb{K}_n]^{-1}\,A\varphi_n(\mathbf{Z})^H. \tag{20}$$

Using Lemma 4, we can express (19) as a ratio of determinants

$$\frac{k_n}{\gamma_n}\,\frac{\psi_n(z)}{\varphi_n(z)} = \frac{\det\left[\,I_M + A\,\mathbb{K}_n - A\,\dfrac{K_n(z,\mathbf{Z})^T}{\varphi_n(z)}\,\varphi_n(\mathbf{Z})\right]}{\det[I_M + A\,\mathbb{K}_n]}. \tag{21}$$

Doing the same with (20), we obtain

$$\left(\frac{k_n}{\gamma_n}\right)^2 = \frac{\det[I_M + A\,\mathbb{K}_n + A\varphi_n(\mathbf{Z})^T\,\varphi_n(\mathbf{Z})]}{\det[I_M + A\,\mathbb{K}_n]}$$

$$\left(\frac{k_n}{\gamma_n}\right)^2 = \frac{\det[I_M + A\,\mathbb{K}_{n+1}]}{\det[I_M + A\,\mathbb{K}_n]}. \tag{22}$$

Formulas (21) and (22) are used in order to obtain the asymptotic behavior of k_n/γ_n and $\psi_n(z)/\varphi_n(z)$ for $|z|>1$.

By assumption A is a positive definite matrix. We can express (22) as

$$\left(\frac{k_n}{\gamma_n}\right)^2 = \frac{\det[A^{-1} + \mathbb{K}_{n+1}]}{\det[A^{-1} + \mathbb{K}_n]}.$$

Now, we will find the asymptotic behavior of k_n/γ_n:

$$\lim_{n\to\infty}\left(\frac{k_n}{\gamma_n}\right)^2 = \lim_{n\to\infty}\frac{\det[A^{-1} + \mathbb{K}_{n+1}]}{\det[A^{-1} + \mathbb{K}_n]}.$$

If we introduce the diagonal matrix

$$\Lambda_n = \operatorname{diag}\left(\frac{1}{\varphi_n(z_1)}, \frac{1}{\varphi_n'(z_1)}, \ldots, \frac{1}{\varphi_n^{(l_1)}(z_1)}, \ldots, \frac{1}{\varphi_n(z_m)}, \frac{1}{\varphi_n'(z_m)}, \ldots, \frac{1}{\varphi_n^{(l_m)}(z_m)}\right),$$

we have

$$\lim_{n\to\infty}\left(\frac{k_n}{\gamma_n}\right)^2$$
$$= \lim_{n\to\infty}\frac{\det[\overline{\Lambda_{n+1}}A^{-1}\Lambda_{n+1} + \overline{\Lambda_{n+1}}\mathbb{K}_{n+1}\Lambda_{n+1}]}{\det[\overline{\Lambda_n}A^{-1}\Lambda_n + \overline{\Lambda_n}\mathbb{K}_n\Lambda_n]}\;\frac{\det[\overline{\Lambda_n}\Lambda_n]}{\det[\overline{\Lambda_{n+1}}\Lambda_{n+1}]}.$$

The matrix $\overline{\Lambda_n}\mathbb{K}_n\Lambda_n$ can be described by blocks. The r, s block is an $(l_r+1)\times(l_s+1)$ matrix

$$\left(\frac{K_n^{(j,i)}(z_s, z_r)}{\varphi_n^{(j)}(z_s)\,\overline{\varphi_n^{(i)}(z_r)}}\right)_{i=0,\dots,l_r}^{j=0,\dots,l_s},$$

where $r, s = 1, \dots, m$. Using Lemma 1 and Lemma 3, we conclude that

$$\lim_{n\to\infty} \det[\overline{\Lambda_{n+1}}A^{-1}\Lambda_{n+1} + \overline{\Lambda_{n+1}}\mathbb{K}_{n+1}\Lambda_{n+1}] = 0,$$

and we need to compute a limit of the form 0/0, which is undetermined. In [3], we find a similar situation for a system of equations. We adapt here some ideas that appear in that work.

For all f, h differentiable functions and $v = 0, 1, 2, \dots$ it holds

$$\frac{f^{(v)}}{h^{(v)}} = \left(\frac{f}{h}\right)^{(v)} \frac{h}{h^{(v)}} - \sum_{k=1}^{v} F(v,k)\,\frac{f^{(v-k)}}{h^{(v-k)}}, \tag{23}$$

where

$$F(v,k) = \binom{v}{k}\frac{hh^{(v-k)}}{h^{(v)}}\left(\frac{1}{h}\right)^{(k)}.$$

Notice that the coefficients $F(v, k)$ do not depend on the function f. If we take $f = h$ we get the relation

$$1 + \sum_{k=1}^{v} F(v,k) = 0. \tag{24}$$

Now, in

$$\det[\overline{\Lambda_{n+1}}A^{-1}\Lambda_{n+1} + \overline{\Lambda_{n+1}}\mathbb{K}_{n+1}\Lambda_{n+1}]$$

add to the $\sum_{p=1}^{s-1}(l_p+1)+1+k$ row, for $1 \leqslant k \leqslant l_s$ and $1 \leqslant s \leqslant m$, a linear combination of the preceding $k-1$ rows with the coefficients defined by (23) with

$$h(z) := \overline{\varphi_{n+1}(z)}$$

and $z = z_s$, then multiply the resulting row by

$$\frac{\overline{h^{(k)}(z)}}{h(z)}$$

evaluated at $z=z_s$. We also carry out this kind of elementary operations by rows with

$$\det[\overline{\Lambda_n} A^{-1}\Lambda_n+\overline{\Lambda_n}\mathbb{K}_n\Lambda_n],$$

where in this case

$$h(z):=\overline{\varphi_n(z)}.$$

On doing these elementary operations by rows we find that

$$\frac{\det[\overline{\Lambda_{n+1}} A^{-1}\Lambda_{n+1}+\overline{\Lambda_{n+1}}\mathbb{K}_{n+1}\Lambda_{n+1}]}{\det[\overline{\Lambda_n} A^{-1}\Lambda_n+\overline{\Lambda_n}\mathbb{K}_n\Lambda_n]} = \frac{\prod_{j=1}^{m}\prod_{s=1}^{l_j}\dfrac{\overline{\varphi_n^{(s)}(z_j)}}{\varphi_n(z_j)}\det[B_{n+1}+H_{n+1}]}{\prod_{j=1}^{m}\prod_{s=1}^{l_j}\dfrac{\overline{\varphi_{n+1}^{(s)}(z_j)}}{\varphi_{n+1}(z_j)}\det[B_n+H_n]}. \tag{25}$$

Here B_n is a matrix which can be described by blocks. The r, s block is the $(l_r+1)\times(l_s+1)$ matrix

$$\left(\left[\frac{b_{i,j}^{r,s}}{\varphi_n^{(j)}(z_s)\,\overline{\varphi_n^{(i)}(z_r)}}+\sum_{k=1}^{i}F(i,k)\frac{b_{i-k,j}^{r,s}}{\varphi_n^{(j)}(z_s)\,\overline{\varphi_n^{(i-k)}(z_r)}}\right]\frac{\overline{\varphi_n^{(i)}(z_r)}}{\varphi_n(z_r)}\right)_{i=0,\ldots,l_r}^{j=0,\ldots,l_s},$$

where $b_{i,j}^{r,s}$ are constants,

$$F(i,k)=\binom{i}{k}\overline{\varphi_n(z_r)\frac{\varphi_n^{(i-k)}(z_r)}{\varphi_n^{(i)}(z_r)}\left(\frac{1}{\varphi_n(z_r)}\right)^{(k)}}. \tag{26}$$

Also H_n is a matrix which can be described by blocks. The r, s block is the $(l_r+1)\times(l_s+1)$ matrix

$$\left(\overline{\frac{\partial^i}{\partial w^i}\frac{\overline{K_n^{(j,0)}(z_s,w)}}{\varphi_n^{(j)}(z_s)\,\overline{\varphi_n(w)}}}\Bigg|_{w=z_r}\right)_{i=0,\ldots,l_r}^{j=0,\ldots,l_s},$$

where $r, s=1, ..., m$. Notice that

$$\overline{\frac{\partial^i}{\partial w^i}\frac{\overline{K_n^{(j,0)}(z_s,w)}}{\varphi_n^{(j)}(z_s)\,\overline{\varphi_n(w)}}}\Bigg|_{w=z_r}=\frac{\overline{\dfrac{\partial^j}{\partial z^j}\dfrac{\partial^i}{\partial w^i}\dfrac{\overline{K_n(z,w)}}{\overline{\varphi_n(w)}}\Bigg|_{w=z_r}}}{\varphi_n^{(j)}(z)}\Bigg|_{z=z_s}$$

for $1\leqslant r,\ s,\ \leqslant m$, $0\leqslant j\leqslant l_s$ and $0\leqslant i\leqslant l_r$.

Before we can find the limit in (25) as n tends to infinity, we have to carry out transformations similar to those above but by columns on the determinants $\det[B_{n+1}+H_{n+1}]$ and $\det[B_n+H_n]$. We describe these elementary operations on $\det[B_{n+1}+H_{n+1}]$. Those corresponding to $\det[B_n+H_n]$ are the same with $n+1$ substituted by n.

Let $1\leqslant k\leqslant l_s$ and $1\leqslant s\leqslant m$. Add to the $\sum_{p=1}^{s-1}(l_p+1)+1+k$ column of $\det[B_{n+1}+H_{n+1}]$ a linear combination of the preceeding $k-1$ columns with the coefficients defined in (23) with

$$h(z):=\varphi_{n+1}(z)$$

evaluated at $z=z_s$ and then multiply the resulting column by

$$\frac{h^{(k)}(z)}{h(z)}$$

evaluated at $z=z_s$.

After carrying out similar operations on $\det[B_n+H_n]$, we find that

$$\frac{\displaystyle\prod_{j=1}^{m}\prod_{s=1}^{l_j}\frac{\overline{\varphi_n^{(s)}(z_j)}}{\varphi_n(z_j)}\det[B_{n+1}+H_{n+1}]}{\displaystyle\prod_{j=1}^{m}\prod_{s=1}^{l_j}\frac{\overline{\varphi_{n+1}^{(s)}(z_j)}}{\varphi_{n+1}(z_j)}\det[B_n+H_n]} =\frac{\displaystyle\prod_{j=1}^{m}\prod_{s=1}^{l_j}\left|\frac{\varphi_n^{(s)}(z_j)}{\varphi_n(z_j)}\right|^2}{\displaystyle\prod_{j=1}^{m}\prod_{s=1}^{l_j}\left|\frac{\varphi_{n+1}^{(s)}(z_j)}{\varphi_{n+1}(z_j)}\right|^2}\,\frac{\det[C_{n+1}+R_{n+1}]}{\det[C_n+R_n]},$$

where C_n is a block matrix. The r, s block is the $(l_r+1)\times(l_s+1)$ matrix whose (i, j) entry for $i=0, ..., l_r$ and $j=0, ..., l_s$ is given by

$$\frac{\varphi_n^{(j)}(z_s)}{\varphi_n(z_s)}\left[\left[\frac{b_{i,j}^{r,s}}{\varphi_n^{(j)}(z_s)\,\overline{\varphi_n^{(i)}(z_r)}}+\sum_{k=1}^{i}F(i,k)\frac{b_{i-k,j}^{r,s}}{\varphi_n^{(j)}(z_s)\,\overline{\varphi_n^{(i-k)}(z_r)}}\right]\frac{\overline{\varphi_n^{(i)}(z_r)}}{\varphi_n(z_r)}\right.$$
$$+\sum_{u=1}^{j}\tilde{F}(j,u)\left[\frac{b_{i,j-u}^{r,s}}{\varphi_n^{(j-u)}(z_s)\,\overline{\varphi_n^{(i)}(z_r)}}+\sum_{k=1}^{i}F(i,k)\frac{b_{i-k,j-u}^{r,s}}{\varphi_n^{(j-u)}(z_s)\,\overline{\varphi_n^{(i-k)}(z_r)}}\right]$$
$$\left.\times\frac{\overline{\varphi_n^{(i)}(z_r)}}{\varphi_n(z_r)}\right],$$

where

$$\tilde{F}(j,u)=\binom{j}{u}\varphi_n(z_s)\frac{\varphi_n^{(j-u)}(z_s)}{\varphi_n^{(j)}(z_s)}\left(\frac{1}{\varphi_n(z_s)}\right)^{(u)},$$

and $F(i, k)$ is given by (26). Notice that the elements of the matrix C_n are $o(1)$, and R_n is a block matrix. The r, s block of R_n is the $(l_r+1)\times(l_s+1)$ matrix

$$\left(\frac{\partial^j}{\partial z^j}\frac{\partial^i}{\partial w^i}\overline{\left[\frac{K_n(z,w)}{\varphi_n(z)\,\overline{\varphi_n(w)}}\right]}\Bigg|_{\substack{z=z_s\\ w=z_r}}\right)_{i=0,\,\ldots,\,l_r}^{j=0,\,\ldots,\,l_s},$$

where $r, s=1, \ldots, m$. Taking into account Lemma 1, we obtain

$$\lim_{n\to\infty}\det[C_n+R_n]=\lim_{n\to\infty}\det[o(1)+R_n]=|F_m|\neq 0,$$

where F_m is the matrix defined in (15) and $|F_m|$ denotes its determinant. From this and using (6), we have

$$\lim_{n\to\infty}\left(\frac{k_n}{\gamma_n}\right)^2=\prod_{\substack{i=1,\,\ldots,\,m\\ j=0,\,\ldots,\,l_i}}|z_i|^2,$$

$$\lim_{n\to\infty}\frac{k_n}{\gamma_n}=\prod_{i=1}^{m}|z_i|^{l_i+1}.$$

Using similar arguments, we can obtain the asymptotic behavior of $\psi_n(z)/\varphi_n(z)$. On account of (21) and (3) this reduces to finding the limit of

$$\frac{\det\left[A^{-1}+\mathbb{K}_n-\dfrac{K_n(z,\mathbf{Z})^T}{\varphi_n(z)}\,\varphi_n(\mathbf{Z})\right]}{\det[A^{-1}+\mathbb{K}_n]}$$

$$=\frac{\det\left[\overline{\Lambda_n}A^{-1}\Lambda_n+\overline{\Lambda_n}\mathbb{K}_n\Lambda_n-\overline{\Lambda_n}\,\dfrac{K_n(z,\mathbf{Z})^T}{\varphi_n(z)}\,\varphi_n(\mathbf{Z})\,\Lambda_n\right]}{\det[\overline{\Lambda_n}A^{-1}\Lambda_n+\overline{\Lambda_n}\mathbb{K}_n\Lambda_n]}.$$

In

$$\det\left[\overline{\Lambda_n}A^{-1}\Lambda_n+\overline{\Lambda_n}\mathbb{K}_n\Lambda_n-\overline{\Lambda_n}\,\frac{K_n(z,\mathbf{Z})^T}{\varphi_n(z)}\,\varphi_n(\mathbf{Z})\,\Lambda_n\right]$$

add to the $\sum_{p=1}^{s-1}(l_p+1)+1+k$ row, for $1\leqslant k\leqslant l_s$ and $1\leqslant s\leqslant m$, a linear combination of the preceding $k-1$ rows with the coefficients defined in (23) with

$$h(z):=\overline{\varphi_n(z)}$$

evaluated at $z=z_s$, and multiply the resulting row by

$$\frac{\overline{h^{(k)}(z)}}{h(z)}$$

evaluated at $z = z_s$. The resulting determinant is transformed by columns in similar form. The same transformations by rows and columns are made on $\det[\overline{\Lambda_n} A^{-1} \Lambda_n + \overline{\Lambda_n} \mathbb{K}_n \Lambda_n]$.

Taking into account (24), we find that

$$\lim_{n \to \infty} \frac{\det \left[\overline{\Lambda_n} A^{-1} \Lambda_n + \overline{\Lambda_n} \mathbb{K}_n \Lambda_n - \overline{\Lambda_n} \dfrac{K_n(z, \mathbf{Z})^T}{\varphi_n(z)} \varphi_n(\mathbf{Z}) \Lambda_n \right]}{\det[\overline{\Lambda_n} A^{-1} \Lambda_n + \overline{\Lambda_n} \mathbb{K}_n \Lambda_n]} = \frac{f(z)}{|F_m|},$$

where F_m is the matrix defined in (15) and $f(z)$ is the determinant of a block matrix whose r, s block is the $(l_r+1) \times (l_s+1)$ matrix whose first column is equal to

$$\begin{pmatrix} g(z_s, \overline{z_r}) - g(z, \overline{z_r}) \\ g^{(0,1)}(z_s, \overline{z_r}) - g^{(0,1)}(z, \overline{z_r}) \\ \vdots \\ g^{(0,l_r)}(z_s, \overline{z_r}) - g^{(0,l_r)}(z, \overline{z_r}) \end{pmatrix}$$

and the rest of this matrix can be described as

$$(g^{(i-1, j-1)}(z_s, \overline{z_r}))_{i=1, \dots, l_r+1}^{j=2, \dots, l_s+1},$$

where $r, s = 1, \dots, m$. If we subtract to the $\sum_{s=1}^{i-1} (l_s+1)+1$ column of $f(z)$ its first column for $i = 2, \dots, m$, we obtain that the dependence on the variable z only appears in the first column of this determinant. From this, if we define

$$p(z) := f(z) \prod_{j=1}^{m} (z\overline{z_j} - 1)^{l_j+1} \tag{27}$$

it follows immediately that p is a polynomial in the variable z of degree at most $\sum_{i=1}^{m} (l_i+1)$. Furthermore,

$$f^{(s)}(z)|_{z=z_i} = 0, \qquad 0 \leqslant s \leqslant l_i, \quad i = 1, \dots, m.$$

This implies that

$$p^{(s)}(z)|_{z=z_i} = 0, \qquad 0 \leqslant s \leqslant l_i, \quad i = 1, \dots, m.$$

From this, we deduce that either p is the polynomial identically equal to zero, or there exists a non-zero constant $C \in \mathbb{C}$ such that

$$p(z) = C \prod_{i=1}^{m} (z - z_i)^{l_i+1}.$$

Let us calculate $p^{(\sum_{i=1}^{m}(l_i+1))}(z)$ using Leibniz's formula on (27). If we take into account the equality

$$\left(\prod_{j=1}^{m}(z\overline{z_j}-1)^{l_j+1}\right)^{(\sum_{i=1}^{m}(l_i+1))}=\left(\sum_{i=1}^{m}(l_i+1)\right)!\prod_{j=1}^{m}\overline{z_j}^{l_j+1}$$

and that

$$\prod_{j=1}^{m}(z\overline{z_j}-1)^{l_j+1}\,g^{(0,s)}(z,\overline{z_i}),\qquad i=1,...,m,\quad s=0,...,l_i$$

is a polynomial in the variable z of degree $\sum_{i=1}^{n}(l_i+1)-1$ (therefore its $\sum_{i=1}^{m}(l_i+1)$ derivative is identically zero), it holds that $p(z)^{(\sum_{i=1}^{m}(l_+1))}$ is equal to

$$\left(\sum_{i=1}^{m}(l_i+1)\right)!\prod_{j=1}^{m}\overline{z_j}^{l_j+1}\,|F_m|\neq 0.$$

Therefore,

$$f(z)=\prod_{i=1}^{m}\left(\frac{\overline{z_i}(z-z_i)}{(z\overline{z_i}-1)}\right)^{l_i+1}|F_m|.$$

From this, (3), and (21), we immediately deduce

$$\lim_{n\to\infty}\frac{\psi_n(z)}{\varphi_n(z)}=\prod_{i=1}^{m}\left(\frac{\overline{z_i}(z-z_i)}{|z_i|\,(z\overline{z_i}-1)}\right)^{l_i+1},$$

which is the same as (2) for $k=0$. For arbitrary k, formula (2) follows by induction on account of the identity

$$\frac{q}{q'}\left(\frac{p}{q}\right)'=\frac{p'}{q'}-\frac{p}{q}.$$

Now, we can also prove that

$$\frac{\psi_n(z)}{\varphi_n^*(z)}\rightrightarrows 0,\qquad |z|<1.$$

In fact, from (19), we have

$$\frac{k_n}{\gamma_n}\frac{\psi_n(z)}{\varphi_n^*(z)}=\frac{\varphi_n(z)}{\varphi_n^*(z)}-\varphi_n(\mathbf{Z})[I_M+A\,\mathbb{K}_n]^{-1}A\,\frac{K_n(z,\mathbf{Z})^T}{\varphi_n^*(z)}.$$

Using Lemma 4, we see that

$$\frac{k_n}{\gamma_n}\frac{\psi_n(z)}{\varphi_n^*(z)}=\frac{\varphi_n(z)}{\varphi_n^*(z)}-1+\frac{\det\left[I_M+A\mathbb{K}_n-A\,\frac{K_n(z,\mathbf{Z})^T}{\varphi_n^*(z)}\,\varphi_n(\mathbf{Z})\right]}{\det[I_M+A\,\mathbb{K}_n]}.$$

Using the same kind of arguments as before and taking into account Lemma 2, we can prove that

$$\frac{\det\left[A^{-1}+\mathbb{K}_n-\frac{K_n(z,\mathbf{Z})^T}{\varphi_n^*(z)}\,\varphi_n(\mathbf{Z})\right]}{\det[A^{-1}+\mathbb{K}_n]}\rightrightarrows 1, \qquad |z|<1.$$

From this the statement of Theorem 1 follows. ∎

Remark. We point out that the matrix F_m defined in (15) is not only a non-singular but a positive definite matrix because we have obtained it as a limit of positive definite matrices.

Remark. Taking into account that (22) is still true when A is positive semidefinite, we can consider A to be a block diagonal matrix of the form

$$A=\begin{pmatrix} A_1 & 0_{l_1+1\times l_2+1} & \cdots & 0_{l_1+1\times l_m+1}\\ 0_{l_2+1\times l_1+1} & A_2 & \cdots & 0_{l_2+1\times l_m+1}\\ \vdots & \vdots & \cdots & \vdots\\ 0_{l_m+1\times l_1+1} & 0_{l_m+1\times l_2+1} & \cdots & A_m\end{pmatrix}, \tag{28}$$

where A_i is an $(l_i+1)\times(l_i+1)$ positive semidefinite matrix of rank n_i for $i, j=1, ..., m$. In this case, we conjecture that

$$\lim_{n\to\infty}\frac{k_n}{\gamma_n}=\prod_{i=1}^m |z_i|^{n_i},$$

$$\frac{\psi_n^{(k)}(z)}{\varphi_n^{(k)}(z)}\rightrightarrows\prod_{i=1}^m\left(\frac{\overline{z_i}(z-z_i)}{|z_i|\,(z\overline{z_i}-1)}\right)^{n_i}, \qquad |z|>1, \quad k=0, 1, ...,$$

and

$$\frac{\psi_n(z)}{\varphi_n^*(z)}\rightrightarrows 0, \qquad |z|<1.$$

An immediate consequence would be

$$\frac{\psi_{n+1}^{(k)}(z)}{\psi_n^{(k)}(z)}\rightrightarrows z, \qquad \frac{\psi_n^{(k+1)}(z)}{n\psi_n^{(k)}(z)}\rightrightarrows\frac{1}{z}$$

on the region $\{z\in\mathbb{C}: |z|>\}\setminus\{z_j\}_{j=1}^m$, for $k=0, 1, ...$

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